

On a Class of Polynomials with Integer Coefficients

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Abstract

A class $P_{n,m,p}(x)$ of polynomials is defined. The combinatorial meaning of its coefficients is given. Chebyshev polynomials are the special cases of $P_{n,m,p}(x)$. It is first shown that $P_{n,m,p}(x)$ may be expressed in terms of $P_{n,0,p}(x)$. From this we derive that $P_{n,2,2}(x)$ may be obtain in terms of trigonometric functions, from which we obtain some of its important properties.

Some questions about orthogonality are also concerned.

Furthermore, it is shown that $P_{n,2,2}(x)$ fulfills the same three terms recurrence as Chebyshev polynomials. Some others recurrences for $P_{n,m,p}(x)$ and its coefficients are also obtained.

At the end a formula for coefficients of Chebyshev polynomials of the second kind is derived.

1 Introduction

In the paper [1] the following result is proved.

Theorem A. *If a finite set X consists of n blocks of the size p and an additional block of the size m then, for $n \geq 0$, $k \geq 0$, the number $f(n, k, m, p)$ of $n + k$ -subsets of X intersecting each block of the size p is*

$$f(n, k, m, p) = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{np + m - ip}{n + k}.$$

The following relations for the function f are also proved in [1]:

$$f(n, k, m, p) = \sum_{i=0}^m \binom{m}{i} f(n, k - i, 0, p), \quad (1)$$

$$f(n, k, m, p) = \sum_{i=0}^t (-1)^i \binom{t}{i} f(n, k + t, m + t - i, p), \quad (2)$$

$$f(n, k, m, p) = \sum_{i=1}^p \binom{p}{i} f(n - 1, k - i + 1, m, p - 1), \quad (3)$$

$$f(n, k, m, p) = \sum_{i=0}^n \sum_{j=0}^i \binom{n}{i} \binom{i}{j} f(n-j, k-i+j, m, p-1). \quad (4)$$

Furthermore, it is shown that $(-1)^k f(n, k, 0, 2)$ is the coefficient of Chebyshev polynomial $U_{n+k}(x)$ by x^{n-k} , and that $(-1)^k f(n, k, 1, 2)$ is the coefficient of Chebyshev polynomial $T_{n+k-1}(x)$ by x^{n-k+1} .

Definition 1.1 We define the set of coefficients

$$\{c(n, k, m, p) : n = m, m+1, \dots; k = 0, 1, \dots, n\}$$

such that

$$c(n, k, m, p) = (-1)^{\frac{n-k}{2}} f\left(\frac{n+k-2m}{2}, \frac{n-k}{2}, m, p\right),$$

if n and k are of the same parity, and $c(n, k, m, p) = 0$ otherwise. Polynomials $P_{n,m,p}(x)$ are defined to be

$$P_{n,m,p}(x) = \sum_{k=0}^n c(n, k, m, p) x^k.$$

Remark 1.1 Chebyshev polynomials are particular cases of $P_{n,m,p}(x)$, obtained for $m = 1$, $p = 2$ and $m = 0$, $p = 2$, that is,

$$U_n(x) = P_{n,0,2}, \quad T_n(x) = P_{n,1,2}.$$

The polynomial $P_{n,2,2}(x)$ is the closest to Chebyshev polynomials, and will be denoted simply by $P_n(x)$.

In the next table we state the first few of $P_n(x)$.

$$\begin{aligned} & x^2 \\ & 2x^3 - 2x \\ & 4x^4 - 5x^2 + 1 \\ & 8x^5 - 12x^3 + 4x \\ & 16x^6 - 28x^4 + 13x - 1 \\ & 32x^7 - 64x^5 + 38x^3 - 6x. \end{aligned}$$

Among coefficients of the above polynomials the following sequences from [2] appear: A024623, A049611, A055585, A001844, A035597.

Triangles of coefficients for $P_{n,m,2}(x)$, ($m = 2, 3, 4, 5, 6$) are given in A136388, A136389, A136390, A136397, A136398 respectively.

2 Reduction to the case $m = 0$.

We shall first prove an analog of the formula (1) for polynomials.

Theorem 2.1 *The following equation is fulfilled:*

$$P_{n,m,p}(x) = \sum_{i=0}^m (-1)^i \binom{m}{i} x^{m-i} P_{n-m-i,0,p}(x).$$

Proof. It holds

$$P_{n,m,p}(x) = \sum_{k=0}^n (-1)^{\frac{n-k}{2}} f\left(\frac{n+k-2m}{2}, \frac{n-k}{2}, m, p\right) x^k.$$

Using (1) one obtains

$$P_{n,m,p}(x) = \sum_{k=0}^n \sum_{i=0}^m (-1)^{\frac{n-k}{2}} \binom{m}{i} f(r, s, 0, p) x^k.$$

where

$$r = \frac{n+k-2m}{2}, \quad s = \frac{n-k}{2} - i.$$

Changing the order of summation yields

$$P_{n,m,p}(x) = \sum_{i=0}^m \binom{m}{i} x^{m-i} \sum_{k=0}^n (-1)^{\frac{n-k}{2}} f(r, s, 0, p) x^{k-m+i}.$$

Terms in the sum on the right side of the preceding equation produce nonzero coefficients only in the case $0 \leq s \leq r$, that is,

$$m - i \leq k \leq n - 2i.$$

It follows that

$$P_{n,m,p}(x) = \sum_{i=0}^m (-1)^i \binom{m}{i} x^{m-i} \sum_{k=m-i}^{n-2i} (-1)^{\frac{n-k}{2}-i} f(r, s, 0, p) x^{k-m+i}.$$

Denoting $k - m + i = j$ we obtain

$$P_{n,m,p}(x) = \sum_{i=0}^m (-1)^i \binom{m}{i} x^{m-i} \sum_{j=0}^{n-i} c(n-i, j, 0, p) x^j,$$

which means that

$$P_{n,m,p}(x) = \sum_{i=0}^m (-1)^i \binom{m}{i} x^{m-i} P(n-m-i, 0, p)(x),$$

and the theorem is proved.

According to the preceding theorem we may express $P_n(x)$ in terms of Chebyshev polynomials of the second kind. Namely, for $m = 2$, $n \geq 4$ holds

$$P_n(x) = x^2 U_{n-2}(x) - 2x U_{n-3}(x) + U_{n-4}(x). \quad (5)$$

This allows us to express $P_n(x)$ in terms of trigonometric functions.

Theorem 2.2 For each $n \geq 3$ holds

$$P_n(\cos \theta) = -\sin \theta \sin(n-1)\theta. \quad (6)$$

Proof. According to (5) and well-known property of Chebyshev polynomials we obtain

$$\sin \theta P_n(\cos \theta) = \cos^2 \theta \sin(n-1)\theta - 2 \cos \theta \sin(n-2)\theta + \sin(n-3)\theta.$$

From the identity

$$2 \cos \theta \sin(n-2)\theta = \sin(n-1)\theta + \sin(n-3)\theta$$

follows

$$\sin \theta P_n(\cos \theta) = \cos^2 \theta \sin(n-1)\theta - \sin(n-1)\theta = -\sin^2 \theta \sin(n-1)\theta.$$

Dividing by $\sin \theta \neq 0$ we prove the theorem.

Note that this proof is valid for $n \geq 4$. The case $n = 3$ may be checked directly.

In the following theorem we prove that $P_n(x)$ have the same important property concerning zeroes as Chebyshev polynomials do.

Theorem 2.3 For $n \geq 3$, the polynomial $P_n(x)$ has all simple zeroes lying in the segment $[-1, 1]$.

Proof. Since

$$U_n(1) = n+1, \quad U_n(-1) = (-1)^n(n+1)$$

the equation (5) implies

$$P_n(1) = U_{n-2}(1) - 2U_{n-3}(1) + U_{n-4}(1) = n-1-2(n-2)+n-3=0,$$

and

$$\begin{aligned} P_n(-1) &= U_{n-2}(-1) + 2U_{n-3}(-1) + U_{n-4}(-1) = \\ &= (-1)^{n-2}(n-1) + 2(-1)^{n-3}(n-2) + (-1)^{n-4}(n-3) = 0. \end{aligned}$$

Thus, $x = -1$ and $x = 1$ are zeroes of $P_n(x)$. The remaining $n-2$ zeroes are obtained from the equation

$$\sin(n-1)\theta = 0,$$

and they are

$$x_k = \cos \frac{k\pi}{n-1}, \quad (k = 1, 2, \dots, n-2).$$

We shall now state an immediate consequence of (6) which shows that values of $P_n(x)$, ($x \in [-1, 1]$) lie inside the unit circle.

Corollary 2.1 For $n \geq 3$ and $x \in [-1, 1]$ we have

$$P_n(x)^2 + x^2 \leq 1.$$

Remark 2.1 Dividing $P_n(x)$ by 2^{n-2} we obtain a polynomial with the leading coefficient 1. Thus, its supremum norm on $[-1, 1]$ is $\leq \frac{1}{2^{n-2}}$, which means that $\frac{1}{2^{n-2}}P_n(x)$ has at most 2 times greater supremum norm, comparing with the supremum norm of $T_n(x)$, that is minimal.

Taking derivative in the equation (6) we obtain the following equation for extreme points of $P_n(x)$:

$$(n-1)\tan\theta + \tan(n-1)\theta = 0.$$

The values $\theta = 0$, and $\theta = \pi$ obviously satisfied this equation, which implies that endpoints $x = -1$ and $x = 1$ are extreme points. The remaining extreme points of $P_3(x)$ are $x = \arctan\sqrt{2}$ and $x = -\arctan\sqrt{2}$.

3 Orthogonality

In this section we investigate the set $\{P_n(x) : n = 2, 3, 4, \dots\}$ concerning to the problem of orthogonality, with respect to some standard Jacobi's weights.

The first result is for the weight $\frac{1}{\sqrt{1-x^2}}$ of Chebyshev polynomials of the first kind.

Theorem 3.1 It holds

$$\int_{-1}^1 \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{4} & m = n \\ -\frac{\pi}{8} & |n-m| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Putting $x = \cos\theta$ implies

$$I = \int_{-1}^1 \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}} dx = \int_0^\pi P_n(\cos\theta)P_m(\cos\theta)d\theta.$$

Using (6) we obtain

$$I = \int_0^\pi \sin^2\theta \sin(n-1)\theta \sin(m-1)\theta d\theta.$$

Transforming the integrating function we obtain

$$\begin{aligned} \sin^2\theta \sin(n-1)\theta \sin(m-1)\theta &= \frac{1}{4} \cos(n-m)\theta - \frac{1}{4} \cos(n+m-2)\theta - \\ &- \frac{1}{8} \cos(n-m-2)\theta - \frac{1}{8} \cos(n-m+2)\theta + \frac{1}{8} \cos(n+m-4)\theta + \frac{1}{8} \cos(n+m)\theta. \end{aligned}$$

Taking into account that $m, n \geq 3$ we conclude that integrals of the terms on the right side of the preceding equation are zero if $n \neq m$ and $|n-m| \neq 2$. If $n = m$ we obtain $I = \frac{\pi}{4}$, and $I = -\frac{\pi}{8}$ if $|n-m| = 2$, and the theorem is proved.

Corollary 3.1 *Each subset of the set $\{P_n(x) : n \geq 3\}$, not containing polynomials $P_k(x)$ and $P_m(x)$ such that $|k - m| = 2$, is orthogonal.*

The next result concerns the weight $\sqrt{1 - x^2}$ of Chebyshev polynomials of the second kind. The result is similar to the result of the preceding theorem.

Theorem 3.2 *It holds*

$$\int_{-1}^1 \sqrt{1 - x^2} P_n(x) P_m(x) dx = \begin{cases} \frac{3\pi}{16} & m = n \\ -\frac{\pi}{8} & |n - m| = 2 \\ \frac{\pi}{32} & |n - m| = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In this case we have

$$\int_{-1}^1 \sqrt{1 - x^2} P_n(x) P_m(x) dx = \int_0^\pi \sin^2 \theta P_n(\cos \theta) P_m(\cos \theta) d\theta.$$

We therefore need to calculate the integral

$$\int_0^\pi \sin^4 \theta \sin(n - 1)\theta \sin(m - 1)\theta d\theta.$$

In this case we have

$$\begin{aligned} \sin^4 \theta \sin(n - 1)\theta \sin(m - 1)\theta &= \frac{3}{16} \cos(n - m)\theta - \frac{3}{16} \cos(n + m - 2)\theta + \\ &+ \frac{1}{32} \cos(n - m - 4)\theta + \frac{1}{32} \cos(n - m + 4)\theta - \frac{1}{32} \cos(n + m - 6)\theta - \frac{1}{32} \cos(n + m + 2)\theta - \\ &- \frac{1}{8} \cos(n - m - 2)\theta - \frac{1}{8} \cos(n - m + 2)\theta + \frac{1}{8} \cos(n + m - 4)\theta + \frac{1}{8} \cos(n + m)\theta. \end{aligned}$$

The integral of each term on the right side with $m \neq n$, $|m - n| \neq 2$, $|n - m| \neq 4$ is zero.

For these particular values we easily obtain the desired result, and the theorem is proved.

Taking, for instance, the weight $(1 - x^2)^{\frac{3}{2}}$ in the similar way one obtains

$$\int_{-1}^1 (1 - x^2)^{\frac{3}{2}} P_n(x) P_m(x) dx = \begin{cases} \frac{5\pi}{32} & m = n \\ -\frac{15\pi}{128} & |n - m| = 2 \\ \frac{3\pi}{64} & |n - m| = 4 \\ -\frac{\pi}{128} & |n - m| = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Considering the weight 1 leads to the following result:

Theorem 3.3 *If m and n are of different parity then*

$$\int_{-1}^1 P_n(x) P_m(x) dx = 0.$$

Proof. In this case we need to calculate the integral

$$\int_0^\pi \sin^3 \theta \sin(n-1)\theta \sin(m-1)\theta d\theta.$$

We have

$$\begin{aligned} \sin^3 \theta \sin(n-1)\theta \sin(m-1)\theta &= -\frac{1}{16} \sin(n-m+3)\theta + \frac{1}{16} \sin(n-m-3)\theta + \\ &+ \frac{1}{16} \sin(n+m+1)\theta - \frac{1}{16} \sin(n+m-5)\theta + \frac{3}{16} \sin(n-m+1)\theta - \\ &- \frac{3}{16} \sin(n-m-1)\theta - \frac{3}{16} \sin(n+m-1)\theta + \frac{1}{16} \sin(n+m-3)\theta. \end{aligned}$$

Since m and n are of different parity each function on the right is of the form $\sin(2k+1)\theta$, which implies that its integral is zero, and the theorem is proved.

4 Some recurrence relations

In this section we prove some recurrence relation for $P_{n,m,p}(x)$ as well as some recurrence relations for their coefficients.

Theorem 4.1 *For each integer $t \geq 0$ holds*

$$P_{n,m,p}(x) = \sum_{i=0}^t (-1)^{t-i} \binom{t}{i} x^i P_{n+2t-i, m+t-i, p}(x).$$

Proof. Translating (2) into the equation for coefficients we obtain

$$c(n, k, m, p) = \sum_{i=0}^t (-1)^{i+t} \binom{t}{i} c(n+2t-i, k-i, m+t-i, p).$$

Multiplying by x^k yields

$$c(n, k, m, p) x^k = \sum_{i=0}^t (-1)^{i+t} \binom{t}{i} x^i c(n+2t-i, k-i, m+t-i, p) x^{k-i},$$

which easily implies the claim of the theorem.

In the case $t = 1$, $m = 1$, $p = 2$ we obtain the following formula, expressing $P_n(x)$ in terms of Chebyshev polynomials of the first kind:

$$P_n(x) = xT_{n-1}(x) - T_{n-2}(x).$$

From this we easily conclude that $P_n(x)$ satisfies the same three term recurrence as Chebyshev polynomials.

Corollary 4.1 *The polynomials $P_{n,m,2}(x)$ satisfy the following equation:*

$$P_{n,m,2}(x) = 2xP_{n-1,m,2}(x) - P_{n-2,m,2}(x),$$

with initial conditions

$$P_{0,m,2}(x) = x^m, \quad P_{1,m,2}(x) = 2x^{m+1} - mx^{m-1}.$$

Combining the equations (1) and (4) we obtain

$$f(n, k, m, p) = \sum_{i=0}^n \sum_{j=0}^i \sum_{t=0}^m \binom{n}{i} \binom{m}{t} \binom{i}{j} f(n-j, k-i+j-t, 0, p-1).$$

Translating this equation into the equation for coefficient we obtain

$$c(n, k, m, p) = \sum_{i=0}^n \sum_{j=0}^i \sum_{t=0}^m \binom{n}{i} \binom{m}{t} \binom{i}{j} (-1)^{i-j+t} c(n-i-t, k+i-2j+t, 0, p-1).$$

Applying the preceding equation several times we obtain the following:

Corollary 4.2 *Coefficients of $P_{n,m,p}(x)$ may be obtained as a functions of coefficients of Chebyshev polynomials of the second kind.*

Converting (3) into the equation for coefficients we obtain

$$c(n, k, m, p) = \sum_{i=1}^p (-1)^i \binom{p}{i} c_{n-i, k+i-2, m, p}.$$

This implies the following:

Corollary 4.3 *Coefficients of $P_{n,m,p}(x)$ may be expressed in terms of coefficients of polynomials $P_{n',m,p}(x)$, where $n' < n$.*

We shall finish the paper with a formula for coefficients of Chebyshev polynomials of the second kind. Taking $p = 2$ in (4) we obtain

$$f(n, k, m, 2) = \sum_{i=0}^n \sum_{j=0}^i \binom{n}{i} \binom{i}{j} f(n-j, k-i+j, m, 1).$$

Since $f(r, s, m, 1) = \binom{m}{s}$ we have

$$f(n, k, m, 2) = \sum_{i=0}^n \sum_{j=0}^i \binom{n}{i} \binom{i}{j} \binom{m}{k-i+j}.$$

For $m = 0$, in the sum on the right side of this equation only terms with $k = i-j$ remains. We thus obtain

$$f(n, k, 0, 2) = \sum_{s=0}^{n-k} \binom{n}{s} \binom{n-s}{k}.$$

Accordingly, the following formula follows

Corollary 4.4 *For coefficients $c(n, k)$ of Chebyshev polynomial $U_n(x)$ hold*

$$c(n, k) = (-1)^{\frac{n-k}{2}} \sum_{i=0}^k \binom{\frac{n+k}{2}}{i} \binom{\frac{n+k}{2} - i}{\frac{n-k}{2}},$$

if n i k are of the same parity and $c(n, k) = 0$ otherwise.

References

- [1] M. Janjic, *An Enumerative Function* , arXiv:0801.1976v2
- [2] N. J. Sloane, *The Encyclopedia of Integer Sequences*, electronically published at www.research.att.com/~njas/sequences/